

3-Regular mixed graphs with optimum Hermitian energy*

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Abstract

Let G be a simple undirected graph, and G^ϕ be a mixed graph of G with the generalized orientation ϕ and Hermitian-adjacency matrix $H(G^\phi)$. Then G is called the underlying graph of G^ϕ . The Hermitian energy of the mixed graph G^ϕ , denoted by $\mathcal{E}_H(G^\phi)$, is defined as the sum of all the singular values of $H(G^\phi)$. A k -regular mixed graph on n vertices having Hermitian energy $n\sqrt{k}$ is called a k -regular optimum Hermitian energy mixed graph. In this paper, we first focus on the problem proposed by Liu and Li [J. Liu, X. Li, Hermitian-adjacency matrices and Hermitian energies of mixed graphs, Linear Algebra Appl. 466(2015), 182–207] of determining all the 3-regular connected optimum Hermitian energy mixed graphs. We then prove that optimum Hermitian energy oriented graphs with underlying graph hypercube are unique (up to switching equivalence).

Keywords: mixed graph, Hermitian energy, Hermitian-adjacency matrix, regular graph

AMS Subject Classification Numbers: 05C20, 05C50, 05C90

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. A generalized orientation ϕ of G is to give each edge of S an orientation according to ϕ , where $S \subseteq E(G)$. Then G^ϕ is called a mixed graph of G with the generalized orientation ϕ . If $S = E(G)$, ϕ is an orientation of G and the mixed graph G^ϕ is an oriented graph. If $S = \emptyset$, then G^ϕ is an undirected graph. Thus we find that mixed graphs incorporate both undirected graphs and oriented graphs as extreme cases. In a mixed graph $G^\phi = (V(G^\phi), E(G^\phi))$, if one element (u, v) in $E(G^\phi)$ is an edge (resp. arc), we denote it by $u \leftrightarrow v$ (resp. $u \rightarrow v$). The graph G is called the underlying graph of G^ϕ . A mixed graph is called regular if its underlying graph is a regular graph. Similarly,

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in terms of defining order, size, degree and so on, we focus only on its underlying graph. For undefined terminology and notations, we refer the reader to [2, 5].

The Hermitian-adjacency matrix $H(G^\phi)$ of G^ϕ with vertex set $V(G^\phi) = \{1, 2, \dots, n\}$ is a square matrix of order n , whose entry h_{kl} is defined as

$$h_{kl} = \begin{cases} h_{lk} = 1, & \text{if } k \leftrightarrow l, \\ -h_{lk} = i, & \text{if } k \rightarrow l, \\ 0, & \text{otherwise,} \end{cases}$$

where i is the imaginary number unit. The spectrum $Sp_H(G^\phi)$ of G^ϕ is defined as the spectrum of $H(G^\phi)$. Since $H(G^\phi)$ is a Hermitian matrix, i.e., $H(G^\phi) = [H(G^\phi)]^* := \overline{[H(G^\phi)]}^T$, the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of $H(G^\phi)$ are all real. In [12], Liu and Li introduced the Hermitian energy of the mixed graph G^ϕ , denoted by $\mathcal{E}_H(G^\phi)$, which is defined as the sum of the singular values of $H(G^\phi)$. Since the singular values of $H(G^\phi)$ are the absolute values of its eigenvalues, we have

$$\mathcal{E}_H(G^\phi) = \sum_{j=1}^n |\lambda_j|.$$

For an oriented graph G^ϕ , Adiga et al. [1] introduced the concept of skew adjacency matrix of G^ϕ , denoted by $S(G^\phi)$, which is defined as $S(G^\phi) = -iH(G^\phi)$. Then, the eigenvalues of $S(G^\phi)$ are $\{-i\lambda_1, -i\lambda_2, \dots, -i\lambda_n\}$. The skew energy of oriented graph G^ϕ is defined by Adiga et al. in [1] as $\mathcal{E}_S(G^\phi) = \sum_{j=1}^n |-i\lambda_j|$. Thus, $\mathcal{E}_H(G^\phi) = \mathcal{E}_S(G^\phi)$, i.e., the Hermitian energy of an oriented graph is equal to its skew energy. For more details about skew energy, we refer the survey [10] to the reader.

Hermitian energy can be viewed as a generalization of the graph energy. The concept of the energy of simple undirected graphs was introduced by Gutman in [8], which is related to the total π -electron energy of the molecule represented by that graph. Since then, the graph energy has been extensively studied. For more details, we refer [11] to the reader.

In [12], Liu and Li gave a sharp upper bound of the Hermitian energy in terms of its order n and the maximum degree Δ , i.e.

$$\mathcal{E}_H(G^\phi) \leq n\sqrt{\Delta}.$$

Furthermore, they showed that the equality holds if and only if $H^2(G^\phi) = \Delta I_n$, which implies that G^ϕ is Δ -regular. For convenience, in this paper a mixed graph on n vertices with maximum degree Δ which satisfies $\mathcal{E}_H(G^\phi) = n\sqrt{\Delta}$ is called an optimum Hermitian energy mixed graph. Let I_n be the identity matrix of order n . For simplicity, we always write I when its order is clear from the context. It is important to determine a family of k -regular mixed graphs with optimum Hermitian energy for any positive integer k . In [12], Liu and Li gave Q_k a suitable generalized orientation such that it has optimum Hermitian energy. Besides, they proposed the following problem:

Problem 1.1 *Determine all the k -regular mixed graphs G^ϕ on n vertices with $\mathcal{E}_H(G^\phi) = n\sqrt{k}$ for each k , $3 \leq k \leq n$.*

Liu and Li [12] showed that a 1-regular connected mixed graph on n vertices has optimum Hermitian energy if and only if it is an edge or arc. At the same time, they also proved that a 2-regular connected mixed graph on n vertices has optimum Hermitian energy if and only if it is one of the three types of mixed 4-cycles. If G_1^ϕ and G_2^ϕ are two k -regular mixed graphs with optimum Hermitian energy, then so is their disjoint union. Thus, we only consider k -regular connected mixed graphs.

In this paper, we firstly characterize all 3-regular connected optimum Hermitian energy mixed graphs. Thus we solve Problem 1.1 for $k = 3$. Afterwards, we prove that optimum Hermitian energy oriented graphs with underlying graph hypercube are unique (up to switching equivalence).

2 Preliminaries

In this section, we give some notations and known results. Besides, we also introduce the definition of switching equivalence.

Let $G = G(V, E)$ be a graph with vertex set V and edge set E . For any $v \in V$, we denote the neighborhood of v by $N_G(v)$ in G . Let $G[S]$ denote the subgraph of G induced by S , where $S \subseteq V$. In addition, we give G a generalized orientation ϕ . Then we get a mixed graph denoted by $G^\phi = (V(G^\phi), E(G^\phi))$ and the Hermitian-adjacency matrix of G^ϕ by $H(G^\phi)$.

In [12], Liu and Li gave a sharp upper bound for the Hermitian energy of a mixed graph and a necessary and sufficient condition to attain the upper bound.

Lemma 2.1 (12, a part of Theorem 3.2). *Let G^ϕ be a mixed graph on n vertices with maximum degree Δ . Then $\mathcal{E}_H(G^\phi) \leq n\sqrt{\Delta}$.*

Lemma 2.2 (12, a part of Corollary 3.3). *Let H be the Hermitian-adjacency matrix of a mixed graph G^ϕ on n vertices. Then $\mathcal{E}_H(G^\phi) = n\sqrt{\Delta}$ if and only if $H^2 = \Delta I_n$ i.e. the inner products $H(u, :) \cdot H(v, :) = 0$, $H(:, u) \cdot H(:, v) = 0$ for different vertices u and v of G^ϕ , where $H(u, :)$ and $H(:, u)$ represent row vector and column vector corresponding to vertex u in $H(G^\phi)$, respectively.*

Moreover, Liu and Li [12] gave a characterization of the k -regular connected optimum Hermitian energy mixed graphs.

Lemma 2.3 (12, a part of Lemma 3.5). *Let G^ϕ be a k -regular connected mixed graph with order n ($n \geq 3$), then $\mathcal{E}_H(G^\phi) = n\sqrt{k}$ if and only if for any pair of vertices u and v with distance not*

more than two in G such that $N(u) \cap N(v) \neq \emptyset$, there are edge-disjoint mixed 4-cycles $uxvy$ of the following three types; see Fig.2.1.

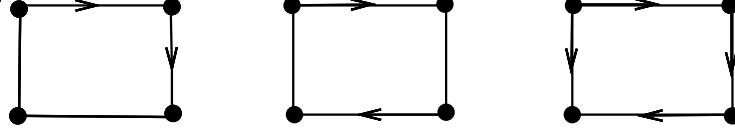


Figure 2.1: Three types of mixed 4-cycles.

By Lemma 2.2, if G^ϕ is a connected mixed graph on n vertices with optimum Hermitian energy $n\sqrt{\Delta}$, then G^ϕ is Δ -regular. Moreover, since any two distinct rows of H are orthogonal, we deduce the following lemma.

Lemma 2.4 *Let H be the Hermitian-adjacency matrix of a k -regular mixed graph G^ϕ on n vertices. If $H^2 = kI_n$, then $|N(u) \cap N(v)|$ is even for any pair of vertices u and v with distance no more than two in G .*

Next we introduce the definition of switching equivalence. Let G^ϕ be a mixed graph with vertex set V . The switching function of G^ϕ is a function $\theta : V \rightarrow T$, where $T = \{1, -1\}$. The switching matrix of G^ϕ is a diagonal matrix $D(\theta) := \text{diag}(\theta(v_k) : v_k \in V)$, where θ is a switching function. Let G^{ϕ_1}, G^{ϕ_2} and G^{ϕ_3} be three mixed graphs with the same underlying graph G and vertex set V . If there exists a switching matrix $D(\theta)$ such that $H(G^{\phi_2}) = D(\theta)^{-1}H(G^{\phi_1})D(\theta)$, then we say G^{ϕ_1} and G^{ϕ_2} are switching equivalent, denoted by $G^{\phi_1} \sim G^{\phi_2}$. If two mixed graphs G^{ϕ_1} and G^{ϕ_2} are switching equivalent, then $Sp_H(G^{\phi_1}) = Sp_H(G^{\phi_2})$ i.e. $\mathcal{E}_H(G^{\phi_1}) = \mathcal{E}_H(G^{\phi_2})$. Besides, the number of arcs (or undirected edges) in G^{ϕ_1} is equal to that in G^{ϕ_2} . Moreover, if $G^{\phi_1} \sim G^{\phi_2}$ and $G^{\phi_2} \sim G^{\phi_3}$, then $G^{\phi_1} \sim G^{\phi_3}$.

Note that Liu and Li [12] also introduced the definition of switching equivalence between mixed graphs. However, T is $\{1, i, -i\}$ in their definition. Besides, our definition coincides with the definition of switching equivalence between oriented graphs which is given in [4] when the mixed graphs are oriented graphs.

Let G^ϕ be a k -regular optimum Hermitian energy mixed graph. If G^ϕ is an oriented graph, then the Hermitian energy of G^ϕ is equal to its skew energy. In [6], Gong and Xu characterized the 3-regular optimum Hermitian energy oriented graphs. Moreover, Chen et al. [3] and Gong et al. [7] independently characterized the 4-regular optimum Hermitian energy oriented graphs. The following lemma is the result about the characterization of 3-regular optimum Hermitian energy oriented graphs in [6].

Lemma 2.5 [6] *Let G^ϕ be a 3-regular optimum Hermitian energy oriented graph. Then G^ϕ (up to isomorphism) is either D_1 or D_2 drawn in Fig.2.2.*

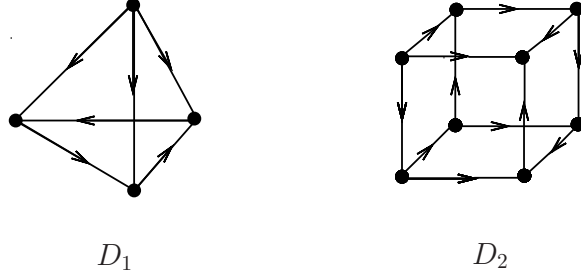


Figure 2.2: 3-regular optimum Hermitian energy oriented graphs.

3 The 3-regular optimum Hermitian energy mixed graphs

In this section, we characterize all 3-regular connected optimum Hermitian energy mixed graphs (up to switching equivalence).

Let G^ϕ be a 3-regular optimum Hermitian energy mixed graph. By Lemma 2.4, we get that G^ϕ satisfies that $|N(u) \cap N(v)|$ is even for any two distinct vertices u and v of G^ϕ . Moreover based on the proof of Theorem 3.5 in [6], we deduce that the underlying graph of G^ϕ with $\mathcal{E}_H(G^\phi) = n\sqrt{3}$ is either the complete graph K_4 or the hypercube Q_3 . Hence, we just need to consider the 3-regular optimum Hermitian energy mixed graphs with underlying graph K_4 or Q_3 .

Firstly, we consider the case that the underlying graph is the complete graph K_4 .

Theorem 3.1 *Let G^ϕ be a 3-regular optimum Hermitian energy mixed graph. If the underlying graph G is K_4 , then G^ϕ is either D_1 drawn in Fig.2.2 or G_1 drawn in Fig.3.3.*

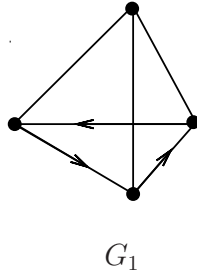


Figure 3.3: Optimum Hermitian energy mixed graph with underlying graph K_4 .

Proof. We divide our discussion into four cases:

Case 1. G^ϕ is an oriented graph.

From Lemma 2.5, we obtain that G^ϕ is D_1 drawn in Fig.2.2.

Case 2. G^ϕ is not an oriented graph and no vertex has two incident edges. Then there exists a vertex, say u_1 , which has one incident edge $u_1 \leftrightarrow u_2$.

Subcase 2.1. $u_1 \rightarrow u_3$ and $u_1 \rightarrow u_4$.

By Lemma 2.2, we have $H(u_1, :) \cdot H(u_2, :) = 0$. Then $\bar{h}_{11}h_{21} + \bar{h}_{12}h_{22} + \bar{h}_{13}h_{23} + \bar{h}_{14}h_{24} = -ih_{23} - ih_{24} = 0$. Hence $h_{23} = i, h_{24} = -i$ or $h_{23} = -i, h_{24} = i$, that is, $u_2 \rightarrow u_3, u_2 \leftarrow u_4$ or $u_2 \leftarrow u_3, u_2 \rightarrow u_4$. Without loss of generality, assume that $u_2 \rightarrow u_3, u_2 \leftarrow u_4$. By Lemma 2.2, it follows that $H(u_1, :) \cdot H(u_3, :) = 0$. Then $\bar{h}_{11}h_{31} + \bar{h}_{12}h_{32} + \bar{h}_{13}h_{33} + \bar{h}_{14}h_{34} = -i - ih_{34} = 0$. Hence $h_{34} = -1$, which is a contradiction.

Subcase 2.2. $u_1 \leftarrow u_3$ and $u_1 \leftarrow u_4$.

By Lemma 2.2, $H(u_1, :) \cdot H(u_2, :) = 0$. Then $\bar{h}_{11}h_{21} + \bar{h}_{12}h_{22} + \bar{h}_{13}h_{23} + \bar{h}_{14}h_{24} = ih_{23} + ih_{24} = 0$. Hence $h_{23} = i, h_{24} = -i$ or $h_{23} = -i, h_{24} = i$, that is, $u_2 \rightarrow u_3, u_2 \leftarrow u_4$ or $u_2 \leftarrow u_3, u_2 \rightarrow u_4$. Without loss of generality, assume that $u_2 \rightarrow u_3, u_2 \leftarrow u_4$. By Lemma 2.2, it follows that $H(u_1, :) \cdot H(u_3, :) = 0$. Then $\bar{h}_{11}h_{31} + \bar{h}_{12}h_{32} + \bar{h}_{13}h_{33} + \bar{h}_{14}h_{34} = -i + ih_{34} = 0$. Hence $h_{34} = 1$ i.e. there is an edge $u_3 \leftrightarrow u_4$ in G^ϕ . However, $H(u_1, :) \cdot H(u_4, :) = \bar{h}_{11}h_{41} + \bar{h}_{12}h_{42} + \bar{h}_{13}h_{43} + \bar{h}_{14}h_{44} = i + i \neq 0$, which is a contradiction.

Subcase 2.3. $u_1 \rightarrow u_3, u_1 \leftarrow u_4$ or $u_1 \leftarrow u_3, u_1 \rightarrow u_4$.

Without loss of generality, assume that $u_1 \rightarrow u_3$ and $u_1 \leftarrow u_4$. By a similar way, we can prove that this subcase could not happen.

Case 3. No vertex has three incident edges, and there exists a vertex, say u_1 , has two incident edges $u_1 \leftrightarrow u_2$ and $u_1 \leftrightarrow u_3$. Then for the vertex u_4 , there is an arc $u_1 \rightarrow u_4$ or $u_1 \leftarrow u_4$.

Suppose that $u_1 \rightarrow u_4$. By Lemma 2.2, we have $H(u_1, :) \cdot H(u_2, :) = 0$. Then $\bar{h}_{11}h_{21} + \bar{h}_{12}h_{22} + \bar{h}_{13}h_{23} + \bar{h}_{14}h_{24} = h_{23} - ih_{24} = 0$. Hence $h_{23} = 1, h_{24} = -i$ or $h_{23} = i, h_{24} = 1$, that is, $u_2 \leftrightarrow u_3, u_2 \leftarrow u_4$ or $u_2 \rightarrow u_3, u_2 \leftrightarrow u_4$. If $u_2 \leftrightarrow u_3, u_2 \leftarrow u_4$, then $H(u_1, :) \cdot H(u_3, :) = 0$ from Lemma 2.2. It implies that $\bar{h}_{11}h_{31} + \bar{h}_{12}h_{32} + \bar{h}_{13}h_{33} + \bar{h}_{14}h_{34} = 1 - ih_{34} = 0$. Thus $h_{34} = -i$ i.e. $u_3 \leftarrow u_4$. However, $H(u_1, :) \cdot H(u_4, :) = \bar{h}_{11}h_{41} + \bar{h}_{12}h_{42} + \bar{h}_{13}h_{43} + \bar{h}_{14}h_{44} = i + i \neq 0$, which is a contradiction. If $u_2 \rightarrow u_3, u_2 \leftrightarrow u_4$, then $H(u_1, :) \cdot H(u_3, :) = 0$ from Lemma 2.2. It implies that $\bar{h}_{11}h_{31} + \bar{h}_{12}h_{32} + \bar{h}_{13}h_{33} + \bar{h}_{14}h_{34} = -i - ih_{34} = 0$. Thus $h_{34} = -1$, which is a contradiction.

For $u_1 \leftarrow u_4$, we can prove that this case could not happen by the similar method.

Case 4. There exists a vertex, say u_1 , has three incident edges $u_1 \leftrightarrow u_2, u_1 \leftrightarrow u_3$ and $u_1 \leftrightarrow u_4$.

Since $H(u_1, :) \cdot H(u_2, :) = 0$, we can obtain that $h_{23} = i, h_{24} = -i$ or $h_{23} = -i, h_{24} = i$, that is $u_2 \rightarrow u_3, u_2 \leftarrow u_4$ or $u_2 \leftarrow u_3, u_2 \rightarrow u_4$. Without loss of generality, assume that $u_2 \rightarrow u_3, u_2 \leftarrow u_4$. Similarly, we have $h_{34} = i$ i.e. $u_3 \rightarrow u_4$ by $H(u_1, :) \cdot H(u_3, :) = 0$. That is $u_2 \rightarrow u_3, u_2 \leftarrow u_4$ and $u_3 \rightarrow u_4$; see G_1 in Fig.3.3.

Thus, the proof is complete. ■

Next, we determine all optimum Hermitian energy mixed graphs with underlying graph Q_3 .

Theorem 3.2 *Let G^ϕ be a 3-regular optimum Hermitian energy mixed graph. If the underlying graph G is Q_3 , then G^ϕ (up to switching equivalence) is one of the following graphs: D_2 or H_i , where $i = 1, 2, \dots, 6$; see Figs. 2.2 and 3.4.*

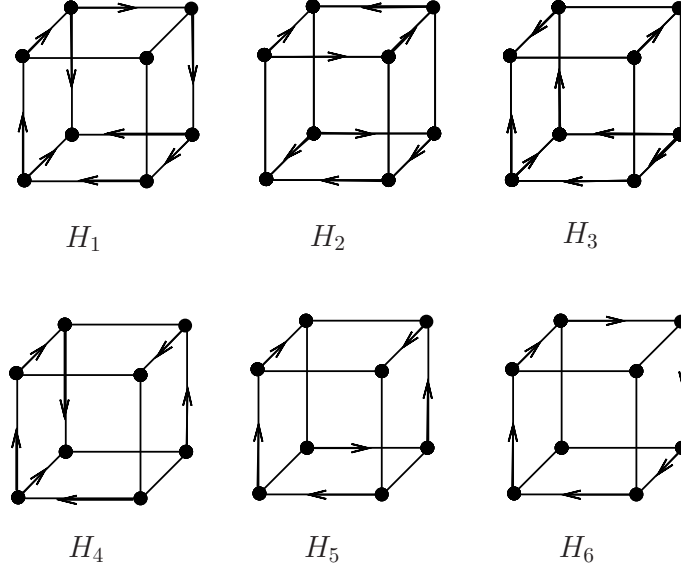


Figure 3.4: Optimum Hermitian energy mixed graphs with underlying graph Q_3 .

Proof. We divide our discussion into two cases:

Case 1. G^ϕ is an oriented graph.

From Lemma 2.5, we obtain that G^ϕ is D_2 drawn in Fig.2.2.

Case 2. G^ϕ is not an oriented graph. In the following, we replace G^ϕ with Q_3^ϕ for convenience and assume that $V(Q_3) = \{v_1, v_2, \dots, v_8\}$, see Fig.3.5.

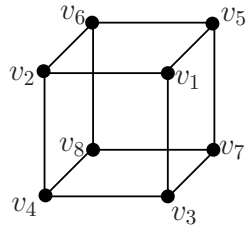


Figure 3.5: Q_3 .

Let a (resp. b) denote the number of arcs (resp. undirected edges) in Q_3^ϕ , where $a + b = 12$. Since Q_3^ϕ is not an oriented graph, we get that $a \leq 11$. Furthermore, there are exactly six mixed 4-cycles in Q_3^ϕ . Let C_1, C_2, C_3, C_4, C_5 and C_6 denote the mixed 4-cycle induced by vertices

$\{v_1, v_2, v_4, v_3\}$, $\{v_2, v_4, v_8, v_6\}$, $\{v_5, v_6, v_8, v_7\}$, $\{v_1, v_3, v_7, v_5\}$, $\{v_1, v_2, v_6, v_5\}$ and $\{v_3, v_4, v_8, v_7\}$, respectively. By Lemma 2.3, we deduce that every mixed 4-cycle in Q_3^ϕ is one of the three types in Fig.2.1. Thus, we obtain the following claim.

Claim 1: In Q_3^ϕ , every mixed 4-cycle has either two arcs and two undirected edges or four arcs.

It follows that each mixed 4-cycle in Q_3^ϕ has at least two arcs. Then, we have $a \geq \frac{2 \times 6}{2} = 6$. Moreover, we check that $a \neq 11, 10$. Consequently, $6 \leq a \leq 9$. Now we divide the discussion about the values of a and b into four subcases:

Subcase 2.1. $a = 9, b = 3$.

In this subcase, we want to determine three undirected edges in Q_3^ϕ . Without loss of generality, suppose that $v_1 \leftrightarrow v_3$. By Claim 1, both mixed 4-cycle C_1 and C_4 have two undirected edges and hence we get the following four cases (up to isomorphism) by considering the other two undirected edges in C_1 and C_4 .

(1) The other two undirected edges are $v_2 \leftrightarrow v_4$ in C_1 and $v_5 \leftrightarrow v_7$ in C_4 . Then, there are three arcs in mixed 4-cycle C_2 and C_3 , which contradicts Claim 1.

(2) The other two undirected edges are $v_1 \leftrightarrow v_2$ in C_1 and $v_5 \leftrightarrow v_7$ in C_4 . Then, there are three arcs in mixed 4-cycle C_5 and C_3 , which contradicts Claim 1.

(3) The other two undirected edges are $v_1 \leftrightarrow v_2$ in C_1 and $v_3 \leftrightarrow v_7$ in C_4 . Then, there are three arcs in mixed 4-cycle C_5 and C_6 , which contradicts Claim 1.

(4) The other two undirected edges are $v_1 \leftrightarrow v_2$ in C_1 and $v_1 \leftrightarrow v_5$ in C_4 . Then, mixed 4-cycle C_1 , C_4 and C_5 should be the first type in Fig.2.1; mixed 4-cycle C_2 , C_3 and C_6 should be the third type in Fig.2.1. Hence, there are two arcs $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ or $v_2 \rightarrow v_4, v_4 \rightarrow v_3$ in C_1 , $v_5 \rightarrow v_7, v_7 \rightarrow v_3$ or $v_3 \rightarrow v_7, v_7 \rightarrow v_5$ in C_4 , and $v_2 \rightarrow v_6, v_6 \rightarrow v_5$ or $v_5 \rightarrow v_6, v_6 \rightarrow v_2$ in C_5 . If there are two arcs $v_3 \rightarrow v_4$ and $v_4 \rightarrow v_2$ in a mixed graph, then we reverse every arc which is incident to vertex v_4 and acquire a new mixed graph. We can prove that the two mixed graphs are switching equivalent by the definition of switching equivalence. Without loss of generality, assume that $v_3 \rightarrow v_4, v_4 \rightarrow v_2$. By a similar discussion, we assume that $v_2 \rightarrow v_6, v_6 \rightarrow v_5$ and $v_5 \rightarrow v_7, v_7 \rightarrow v_3$. Afterwards, we have either $v_4 \rightarrow v_8, v_6 \rightarrow v_8$ or $v_8 \rightarrow v_6, v_8 \rightarrow v_4$ in C_2 . Analogously by switching equivalence, we assume that $v_4 \rightarrow v_8, v_6 \rightarrow v_8$ and then $v_7 \rightarrow v_8$. Therefore, we get the graph (up to switching equivalence) H_1 in Fig.3.4.

Subcase 2.2. $a = 8, b = 4$.

Now we want to determine four undirected edges in Q_3^ϕ . Based on the discussion of subcase 2.1, we just need to find one more undirected edge.

If the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_2 \leftrightarrow v_4$ and $v_5 \leftrightarrow v_7$, then there are two undirected edges in C_1 and C_4 . Thus, the fourth undirected edge cannot be $v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_4, v_1 \leftrightarrow v_5$ or $v_3 \leftrightarrow v_7$. If the fourth undirected edge is $v_2 \leftrightarrow v_6, v_5 \leftrightarrow v_6, v_4 \leftrightarrow v_8$

or $v_7 \leftrightarrow v_8$, then the resulting graphs are isomorphic. Without loss of generality, suppose that $v_2 \leftrightarrow v_6$. Nevertheless, there are three arcs in C_5 , which contradicts Claim 1. If the fourth undirected edge is $v_6 \leftrightarrow v_8$, then mixed 4-cycle C_5 and C_6 should be the third type in Fig.2.1 and the others should be the second type in Fig.2.1. Thus, we get H_2 (up to isomorphism) in Fig.3.4.

If the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_5 \leftrightarrow v_7$, then there are two undirected edges in C_1 and C_4 . Thus, the fourth undirected edge cannot be $v_2 \leftrightarrow v_4, v_3 \leftrightarrow v_4, v_1 \leftrightarrow v_5$ or $v_3 \leftrightarrow v_7$. If the fourth undirected edge is $v_2 \leftrightarrow v_6$, then there are three arcs in C_2 , which contradicts Claim 1. By a similar way, we deduce that the fourth undirected edge cannot be $v_6 \leftrightarrow v_8, v_4 \leftrightarrow v_8$ or $v_7 \leftrightarrow v_8$. If the fourth undirected edge is $v_5 \leftrightarrow v_6$, then mixed 4-cycle C_1 and C_3 should be the first type in Fig.2.1; mixed 4-cycle C_4 and C_5 should be the second type in Fig.2.1; mixed 4-cycle C_2 and C_6 should be the third type in Fig.2.1. Hence, there are two arcs $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ or $v_2 \rightarrow v_4, v_4 \rightarrow v_3$ in C_1 , and $v_7 \rightarrow v_8, v_8 \rightarrow v_6$ or $v_6 \rightarrow v_8, v_8 \rightarrow v_7$ in C_3 . If there are two arcs $v_3 \rightarrow v_4$ and $v_4 \rightarrow v_2$ in a mixed graph, then we reverse every arc which is incident to vertex v_4 and acquire a new mixed graph. We can prove that the two mixed graphs are switching equivalent by the definition of switching equivalence. Without loss of generality, assume that $v_3 \rightarrow v_4, v_4 \rightarrow v_2$. By a similar discussion, we assume that $v_7 \rightarrow v_8, v_8 \rightarrow v_6$. Afterwards, we have either $v_4 \rightarrow v_8$ or $v_8 \rightarrow v_4$. If there is an arc $v_4 \rightarrow v_8$, then we get the other arcs $v_6 \rightarrow v_2, v_1 \rightarrow v_5, v_7 \rightarrow v_3$. Thus, we obtain H_3 depicted in Fig.3.4. If there is an arc $v_8 \rightarrow v_4$, then we get the other arcs $v_2 \rightarrow v_6, v_5 \rightarrow v_1, v_3 \rightarrow v_7$ and the resulting mixed graph is isomorphic to H_3 .

If the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_3 \leftrightarrow v_7$, then there are two undirected edges in C_1 and C_4 . Thus, the fourth undirected edge cannot be $v_2 \leftrightarrow v_4, v_3 \leftrightarrow v_4, v_1 \leftrightarrow v_5$ or $v_5 \leftrightarrow v_7$. If the fourth undirected edge is $v_2 \leftrightarrow v_6$, then there are three arcs in C_2 , which contradicts Claim 1. By a similar way, we deduce that the fourth undirected edge cannot be $v_5 \leftrightarrow v_6, v_6 \leftrightarrow v_8, v_4 \leftrightarrow v_8$ or $v_7 \leftrightarrow v_8$. Thus, this case could not happen.

If the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_1 \leftrightarrow v_5$, then there are two undirected edges in C_1, C_4 and C_5 . Thus, the fourth undirected edge cannot be $v_2 \leftrightarrow v_4, v_3 \leftrightarrow v_4, v_3 \leftrightarrow v_7, v_5 \leftrightarrow v_7, v_2 \leftrightarrow v_6$ or $v_5 \leftrightarrow v_6$. If the fourth undirected edge is $v_4 \leftrightarrow v_8$, then there are three arcs in C_2 , which contradicts Claim 1. By a similar way, we deduce that the fourth undirected edge cannot be $v_6 \leftrightarrow v_8$ or $v_7 \leftrightarrow v_8$. Thus, this case could not happen.

Subcase 2.3. $a = 7, b = 5$.

Similarly in order to determine five undirected edges in Q_3^ϕ , we just need to find two more undirected edges based on the discussion of subcase 2.1.

If the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_2 \leftrightarrow v_4$ and $v_5 \leftrightarrow v_7$, then there are two undirected edges in C_1 and C_4 . Thus, the other two undirected edges cannot

be $v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_4, v_1 \leftrightarrow v_5$ or $v_3 \leftrightarrow v_7$. If one of the other two undirected edges is $v_6 \leftrightarrow v_8$, then there are two undirected edges in C_2 and C_3 . By Claim 1, the last undirected edge cannot be $v_2 \leftrightarrow v_6, v_4 \leftrightarrow v_8, v_5 \leftrightarrow v_6$ or $v_7 \leftrightarrow v_8$. Then, there do not exist five undirected edges and hence this case could not happen. If there is an arc between v_6 and v_8 , then the other two undirected edges (up to isomorphism) can be $v_2 \leftrightarrow v_6$ and $v_4 \leftrightarrow v_8$, $v_2 \leftrightarrow v_6$ and $v_5 \leftrightarrow v_6$, or $v_2 \leftrightarrow v_6$ and $v_7 \leftrightarrow v_8$. If the other two undirected edges are $v_2 \leftrightarrow v_6$ and $v_4 \leftrightarrow v_8$, then there are three undirected edges in C_2 , which contradicts Claim 1. By a similar way, we deduce that the other two undirected edges cannot be $v_2 \leftrightarrow v_6$ and $v_7 \leftrightarrow v_8$. Therefore, the five undirected edges in Q_3^ϕ can be $v_1 \leftrightarrow v_3, v_2 \leftrightarrow v_4, v_5 \leftrightarrow v_7, v_2 \leftrightarrow v_6$ and $v_5 \leftrightarrow v_6$.

By a similar discussion, if the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_5 \leftrightarrow v_7$, then we deduce that the five undirected edges in Q_3^ϕ can be $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_5 \leftrightarrow v_7, v_2 \leftrightarrow v_6$ and $v_6 \leftrightarrow v_8$; if the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_3 \leftrightarrow v_7$, then we deduce that the five undirected edges in Q_3^ϕ can be either $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_2 \leftrightarrow v_6, v_4 \leftrightarrow v_8$ or $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_5 \leftrightarrow v_6, v_7 \leftrightarrow v_8$; if the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_1 \leftrightarrow v_5$, then this case could not happen. Regardless of the labels of vertices, the cases of the five undirected edges which we have determined are the same.

Without loss of generality, suppose that the five undirected edges are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_5 \leftrightarrow v_6$ and $v_7 \leftrightarrow v_8$. Then, mixed 4-cycle C_1, C_4 and C_6 should be the first type in Fig.2.1; mixed 4-cycle C_5 and C_3 should be the second type in Fig.2.1; mixed 4-cycle C_2 should be the third type in Fig.2.1. Hence, there are two arcs either $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ or $v_2 \rightarrow v_4, v_4 \rightarrow v_3$ in C_1 . If there are two arcs $v_3 \rightarrow v_4$ and $v_4 \rightarrow v_2$ in C_1 , then we have an arc $v_4 \rightarrow v_8$ in C_6 . Otherwise, there is an arc $v_8 \rightarrow v_4$. If there are three arcs $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ and $v_4 \rightarrow v_8$ in a mixed graph, then we reverse every arc which is incident to vertex v_4 and acquire a new mixed graph. We can prove that the two mixed graphs are switching equivalent by the definition of switching equivalence. Without loss of generality, assume that $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ and $v_4 \rightarrow v_8$. Afterwards, we have arcs either $v_7 \rightarrow v_5, v_5 \rightarrow v_1$ or $v_1 \rightarrow v_5, v_5 \rightarrow v_7$ in C_4 . If there are two arcs $v_7 \rightarrow v_5$ and $v_5 \rightarrow v_1$ in C_4 , then the other arcs are $v_2 \rightarrow v_6$ and $v_6 \rightarrow v_8$. Thus, we obtain H_4 depicted in Fig.3.4. If there are two arcs $v_1 \rightarrow v_5$ and $v_5 \rightarrow v_7$ in C_4 , then the other arcs are $v_8 \rightarrow v_6, v_6 \rightarrow v_2$ and the resulting mixed graph is isomorphic to H_4 .

Subcase 2.4. $a = 6, b = 6$.

In order to determine six undirected edges in Q_3^ϕ , we just need to find three more undirected edges based on the discussion of subcase 2.1.

If the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_2 \leftrightarrow v_4$ and $v_5 \leftrightarrow v_7$, then there are two undirected edges in C_1 and C_4 . Thus, the other three undirected edges cannot be $v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_4, v_1 \leftrightarrow v_5$ or $v_3 \leftrightarrow v_7$. If one of the other three undirected edges is $v_2 \leftrightarrow v_6$, then there must have an undirected edge $v_5 \leftrightarrow v_6$ in C_5 by Claim 1. However, the last undirected edge cannot be $v_6 \leftrightarrow v_8, v_4 \leftrightarrow v_8$ or $v_7 \leftrightarrow v_8$ by Claim 1. Then, there do not exist six

undirected edges. Similarly, we can show that one of the other three undirected edges cannot be $v_5 \leftrightarrow v_6, v_4 \leftrightarrow v_8$ or $v_7 \leftrightarrow v_8$. Hence, this case could not happen.

By a similar discussion, if the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_5 \leftrightarrow v_7$, then we deduce that the six undirected edges in Q_3^ϕ can be $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_5 \leftrightarrow v_7, v_2 \leftrightarrow v_6, v_4 \leftrightarrow v_8$ and $v_7 \leftrightarrow v_8$; if the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_3 \leftrightarrow v_7$, then we deduce that the six undirected edges in Q_3^ϕ can be either $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_2 \leftrightarrow v_6, v_6 \leftrightarrow v_8, v_7 \leftrightarrow v_8$ or $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_5 \leftrightarrow v_6, v_6 \leftrightarrow v_8, v_4 \leftrightarrow v_8$; if the three undirected edges which we have determined are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2$ and $v_1 \leftrightarrow v_5$, then we deduce that the six undirected edges in Q_3^ϕ can be $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_1 \leftrightarrow v_5, v_6 \leftrightarrow v_8, v_4 \leftrightarrow v_8$ and $v_7 \leftrightarrow v_8$. Regardless of the labels of vertices, the case that $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_5 \leftrightarrow v_7, v_2 \leftrightarrow v_6, v_4 \leftrightarrow v_8, v_7 \leftrightarrow v_8$ is the same with the case that $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_5 \leftrightarrow v_6, v_6 \leftrightarrow v_8, v_4 \leftrightarrow v_8$. Thus, we get the following three cases.

(1) The six undirected edges are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_2 \leftrightarrow v_6, v_6 \leftrightarrow v_8$ and $v_7 \leftrightarrow v_8$. Then, every mixed 4-cycle in Q_3^ϕ should be the first type in Fig.2.1. Hence, there are two arcs either $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ or $v_2 \rightarrow v_4, v_4 \rightarrow v_3$ in C_1 . If there are two arcs $v_3 \rightarrow v_4$ and $v_4 \rightarrow v_2$ in C_1 , then we get an arc $v_4 \rightarrow v_8$ in C_6 and $v_8 \rightarrow v_4$ in C_2 , a contradiction. Analogously, the case that there are two arcs $v_2 \rightarrow v_4$ and $v_4 \rightarrow v_3$ in C_1 could not happen.

(2) The six undirected edges are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_7, v_5 \leftrightarrow v_6, v_6 \leftrightarrow v_8$ and $v_4 \leftrightarrow v_8$. Then, mixed 4-cycle C_1, C_2, C_3 and C_4 should be the first type in Fig.2.1; mixed 4-cycle C_5 and C_6 should be the second type in Fig.2.1. Hence, there are two arcs either $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ or $v_2 \rightarrow v_4, v_4 \rightarrow v_3$ in C_1 . If there are two arcs $v_3 \rightarrow v_4$ and $v_4 \rightarrow v_2$ in C_1 , then we get the other arcs $v_2 \rightarrow v_6, v_5 \rightarrow v_1, v_7 \rightarrow v_5$ and $v_8 \rightarrow v_7$. Thus, we obtain H_5 depicted in Fig.3.4. If there are two arcs $v_2 \rightarrow v_4$ and $v_4 \rightarrow v_3$ in C_1 , then we get the other arcs $v_6 \rightarrow v_2, v_1 \rightarrow v_5, v_5 \rightarrow v_7, v_7 \rightarrow v_8$ and the resulting mixed graph is isomorphic to H_5 .

(3) The six undirected edges are $v_1 \leftrightarrow v_3, v_1 \leftrightarrow v_2, v_1 \leftrightarrow v_5, v_6 \leftrightarrow v_8, v_4 \leftrightarrow v_8$ and $v_7 \leftrightarrow v_8$. Then, every mixed 4-cycle in Q_3^ϕ should be the first type in Fig.2.1. Hence, there are two arcs either $v_3 \rightarrow v_4, v_4 \rightarrow v_2$ or $v_2 \rightarrow v_4, v_4 \rightarrow v_3$ in C_1 . If there are two arcs $v_3 \rightarrow v_4$ and $v_4 \rightarrow v_2$ in C_1 , then we get the other arcs $v_2 \rightarrow v_6, v_6 \rightarrow v_5, v_5 \rightarrow v_7$ and $v_7 \rightarrow v_3$. Thus, we obtain H_6 depicted in Fig.3.4. If there are two arcs $v_2 \rightarrow v_4$ and $v_4 \rightarrow v_3$ in C_1 , then we get the other arcs $v_6 \rightarrow v_2, v_5 \rightarrow v_6, v_7 \rightarrow v_5, v_3 \rightarrow v_7$ and the resulting mixed graph is isomorphic to H_6 .

Thus, the proof is complete. ■

4 The uniqueness of oriented graph Q_k^ϕ with optimum Hermitian energy

It is difficult to determine all optimum Hermitian energy mixed graphs with underlying graph hypercube Q_k for $k \geq 4$. In [13], Tian gave Q_k an orientation such that it has optimum

Hermitian energy. Besides, Gong and Xu [6] proved that 3-regular optimum Hermitian energy oriented graph Q_3^ϕ is unique (up to switching equivalence). In this section, we show that any optimum Hermitian energy oriented graph with underlying graph Q_k is unique (up to switching equivalence) for any positive integer k .

Firstly, we give the following definition about hypercube Q_k which can be found in [9].

Definition 4.1 [9] *A hypercube Q_k of dimension k is defined recursively in terms of the Cartesian product of graphs as follows*

$$Q_k = \begin{cases} K_2, & k = 1, \\ Q_{k-1} \square Q_1, & k \geq 2. \end{cases}$$

Lemma 4.2 *Let Q_k^ϕ be an oriented graph with the orientation ϕ . Then Q_k^ϕ has optimum Hermitian energy if and only if every mixed 4-cycle in Q_k^ϕ is the third type in Fig.2.1.*

Proof. For any two distinct vertices u and v of Q_k , we know that $|N(u) \cap N(v)|$ is either zero or two, where $N(\cdot)$ stands for the neighborhood of a vertex in Q_k . Thus, if there is one common neighbor between the two vertices in Q_k^ϕ , then they have exactly two common neighbors x and y i.e. there is exactly one mixed 4-cycle $uxvy$. By Lemma 2.3, it is easy to obtain this lemma. ■

Hypercube Q_k is a very important family of graphs and it has many nice properties. Liu and Li [12] gave Q_k a suitable orientation such that it has optimum Hermitian energy. Now we give Q_k a new orientation ϕ_0 . For convenience, we assume that the vertex set of Q_k is $\{1, 2, \dots, 2^{k-1}, 2^{k-1}+1, 2^{k-1}+2, \dots, 2^k\}$ with $G[V_1] = G[V_2] = Q_{k-1}$, where $V_1 = \{1, 2, \dots, 2^{k-1}\}$, $V_2 = \{2^{k-1}+1, 2^{k-1}+2, \dots, 2^k\}$. Firstly, we give the hypercube Q_1 an orientation $Q_1^{\phi_0}$ such that $1 \rightarrow 2$. Afterwards, we suppose that Q_{k-1} has been oriented into $Q_{k-1}^{\phi_0}$. By reversing every arc of $Q_{k-1}^{\phi_0}$, we can get another new orientation denoted by $-\phi_0$. For Q_k , we give $G[V_1]$ the orientation ϕ_0 and $G[V_2]$ the orientation $-\phi_0$. Next we put an arc from each vertex in $G[V_1]$ to the corresponding vertex in $G[V_2]$ i.e. $t \rightarrow 2^{k-1} + t$ for $t = 1, 2, \dots, 2^{k-1}$. Then we get $Q_k^{\phi_0}$.

The lemma below shows that the Hermitian energy of $Q_k^{\phi_0}$ is optimum.

Lemma 4.3 *Let Q_k be a hypercube of dimension k with $n = 2^k$ vertices. Then $Q_k^{\phi_0}$ satisfies $H^2(Q_k^{\phi_0}) = kI_n$ (or $\mathcal{E}_H(Q_k^{\phi_0}) = n\sqrt{k}$).*

Proof. If $k = 1$, then

$$H(Q_1^{\phi_0}) = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Hence it is easy to show that $H^2(Q_1^{\phi_0}) = I_2$ and we only need to consider the case $k \geq 2$.

By Lemma 4.2, if every mixed 4-cycle of $Q_k^{\phi_0}$ is the third type in Fig.2.1, then $\mathcal{E}_H(Q_k^{\phi_0}) = n\sqrt{k}$. Therefore, we just need to show that every mixed 4-cycle of $Q_k^{\phi_0}$ is the third type in

Fig.2.1 for $k \geq 2$. We shall apply induction on k . If $k = 2$, $Q_2^{\phi_0}$ is the third type in Fig.2.1. Suppose now that $k > 2$ and the lemma holds for fewer k . Then every mixed 4-cycle of $Q_{k-1}^{\phi_0}$ is the third type in Fig.2.1 and so is $Q_{k-1}^{-\phi_0}$. Moreover by the definition of $Q_k^{\phi_0}$, we have

$$H(Q_k^{\phi_0}) = \begin{bmatrix} H(Q_{k-1}^{\phi_0}) & iI \\ -iI & H(Q_{k-1}^{-\phi_0}) \end{bmatrix}.$$

For any mixed 4-cycle of $Q_k^{\phi_0}$, if it is contained by the induced subgraph $G[V_1]$ or $G[V_2]$, then we get that it is the third type in Fig.2.1 by the induction hypothesis. Thus, we just talk about the mixed 4-cycle C induced by $\{s, t, t + 2^{k-1}, s + 2^{k-1}\}$, where $1 \leq s, t \leq 2^{k-1}$ and $s \neq t$. Without loss of generality, suppose that there is an arc $s \rightarrow t$ in $Q_k^{\phi_0}$. Then, we have arcs $s + 2^{k-1} \leftarrow t + 2^{k-1}$, $s \rightarrow s + 2^{k-1}$ and $t \rightarrow t + 2^{k-1}$. It is easy to see that the mixed 4-cycle C is the third type in Fig.2.1. Above all, we get that every mixed 4-cycle of $Q_k^{\phi_0}$ is the third type in Fig.2.1 for $k \geq 2$. Thus, We complete the proof. \blacksquare

Theorem 4.4 *Let Q_k^ϕ be an optimum Hermitian energy mixed graph with underlying graph Q_k . If Q_k^ϕ is an oriented graph, then Q_k^ϕ is unique (up to switching equivalence) for any positive integer k .*

Proof. We shall apply induction on k . If $k = 1$, Q_k^ϕ is an arc. Thus, we obtain that Q_k^ϕ is unique (up to switching equivalence) for $k = 1$. Now we assume that the theorem holds for fewer k .

Let Q_k^ϕ be an oriented graph with optimum Hermitian energy. For the sake of convenience, assume that $V(Q_k^\phi) = \{1, 2, \dots, 2^{k-1}, 2^{k-1} + 1, 2^{k-1} + 2, \dots, 2^k\}$ such that both $G[V_1]$ and $G[V_2]$ are mixed graphs with underlying graph Q_{k-1} , where $V_1 = \{1, 2, \dots, 2^{k-1}\}$, $V_2 = \{2^{k-1} + 1, 2^{k-1} + 2, \dots, 2^k\}$. From Lemma 4.2, we know that every mixed 4-cycle in Q_k^ϕ is the third type in Fig.2.1. Then every mixed 4-cycle in $G[V_1]$ and $G[V_2]$ is the third type in Fig.2.1. By Lemma 4.2, $G[V_1]$ and $G[V_2]$ have optimum Hermitian energy. Thus, we have

$$H(Q_k^\phi) = \begin{bmatrix} H(G[V_1]) & S \\ S^* & H(G[V_2]) \end{bmatrix},$$

where S is a diagonal matrix and each diagonal element belongs to $\{i, -i\}$. By the induction hypothesis, we can find a switching matrix $D_1(\theta)$ such that $H(Q_{k-1}^{\phi_0}) = D_1^{-1}(\theta)H(G[V_1])D_1(\theta)$. Assume that Q_{k-1} is a bipartite graph with bipartition X and Y . Let a diagonal matrix $D_3(\theta) = \text{diag}(\theta(v_k) | \theta(v_k) = 1, \text{ if } v_k \in X; \theta(v_k) = -1, \text{ if } v_k \in Y)$. Then $H(Q_{k-1}^{-\phi_0}) = D_3^{-1}(\theta)H(Q_{k-1}^{\phi_0})D_3(\theta)$. Hence $Q_{k-1}^{-\phi_0} \sim Q_{k-1}^{\phi_0}$ i.e. $Q_{k-1}^{\phi_0}$ has optimum Hermitian energy. Similarly, we can find the switching matrix $D_2(\theta)$ such that $H(Q_{k-1}^{-\phi_0}) = D_2^{-1}(\theta)H(G[V_2])D_2(\theta)$ by the induction hypothesis.

Let

$$T_1 = \begin{bmatrix} D_1(\theta) & 0 \\ 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & 0 \\ 0 & D_2(\theta) \end{bmatrix}.$$

Then T_1 and T_2 are switching matrices and we get that

$$\begin{aligned}
& T_2^{-1}T_1^{-1}H(Q_k^\phi)T_1T_2 \\
&= \begin{bmatrix} I & 0 \\ 0 & D_2(\theta) \end{bmatrix}^{-1} \begin{bmatrix} D_1(\theta) & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} H(G[V_1]) & S \\ S^* & H(G[V_2]) \end{bmatrix} \begin{bmatrix} D_1(\theta) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D_2(\theta) \end{bmatrix} \\
&= \begin{bmatrix} D_1(\theta)^{-1}H(G[V_1])D_1(\theta) & D_1(\theta)^{-1}SD_2(\theta) \\ D_2(\theta)^{-1}S^*D_1(\theta) & D_2(\theta)^{-1}H(G[V_2])D_2(\theta) \end{bmatrix} \\
&= \begin{bmatrix} H(Q_{k-1}^{\phi_0}) & S_1 \\ S_1^* & H(Q_{k-1}^{-\phi_0}) \end{bmatrix},
\end{aligned}$$

where $S_1 = D_1^{-1}(\theta)SD_2(\theta) = \begin{bmatrix} s_{11} & 0 & \cdots & 0 \\ 0 & s_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{2^{k-1}2^{k-1}} \end{bmatrix}$ and $s_{tt} \in \{i, -i\}$ with $1 \leq t \leq 2^{k-1}$.

Now we divide the discussion about the value of s_{11} into two cases:

Case 1. $s_{11} = i$.

Let $T_3 = I$. Then

$$H = T_3^{-1}T_2^{-1}T_1^{-1}H(Q_k^\phi)T_1T_2T_3 = \begin{bmatrix} H(Q_{k-1}^{\phi_0}) & S_2 \\ S_2^* & H(Q_{k-1}^{-\phi_0}) \end{bmatrix},$$

where $S_2 = I^{-1}S_1(I) = S_1$.

Case 2. $s_{11} = -i$.

Let

$$T_3 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Then

$$H = T_3^{-1}T_2^{-1}T_1^{-1}H(Q_k^\phi)T_1T_2T_3 = \begin{bmatrix} H(Q_{k-1}^{\phi_0}) & S_2 \\ S_2^* & H(Q_{k-1}^{-\phi_0}) \end{bmatrix},$$

where $S_2 = I^{-1}S_1(-I) = -S_1$.

Assume that $S_2 = \begin{bmatrix} s'_{11} & 0 & \cdots & 0 \\ 0 & s'_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s'_{2^{k-1}2^{k-1}} \end{bmatrix}$ and $H = (h_{kl})$. Then $h_{t(2^{k-1}+t)} = s'_{tt}, 1 \leq$

$t \leq 2^{k-1}$ and $h_{1(2^{k-1}+1)} = s'_{11} = i$. Let $T = T_1 T_2 T_3$. Then $H = T^{-1} H(Q_k^\phi) T$, where T is a diagonal matrix and every diagonal element belongs to $\{1, -1\}$. Since $H^2(Q_k^\phi) = kI$, we get that $HH^* = H^2 = kI$. Thus, the inner product of any two rows of H is zero. Suppose that vertex j is a neighbor of vertex 1, where $j \in V_1$. Next we consider the inner product of the first row and the $2^{k-1} + j$ th row in H . If $1 \rightarrow j$, then $h_{1j} = i, h_{(2^{k-1}+1)(2^{k-1}+j)} = -i$ and $h_{(2^{k-1}+j)(2^{k-1}+1)} = \bar{h}_{(2^{k-1}+1)(2^{k-1}+j)} = i$. Since $H(1, :) \cdot H(2^{k-1} + j, :) = h_{1j} \bar{h}_{(2^{k-1}+j)j} + h_{1(2^{k-1}+1)} \bar{h}_{(2^{k-1}+j)(2^{k-1}+1)} = ih_{j(2^{k-1}+j)} + i(-i) = 0$, we get that $h_{j(2^{k-1}+j)} = i$, i.e., $j \rightarrow 2^{k-1} + j$; see Fig.4.6(a). If $1 \leftarrow j$, then $h_{1j} = -i, h_{(2^{k-1}+1)(2^{k-1}+j)} = i$ and $h_{(2^{k-1}+j)(2^{k-1}+1)} = \bar{h}_{(2^{k-1}+1)(2^{k-1}+j)} = -i$. Since $H(1, :) \cdot H(2^{k-1} + j, :) = h_{1j} \bar{h}_{(2^{k-1}+j)j} + h_{1(2^{k-1}+1)} \bar{h}_{(2^{k-1}+j)(2^{k-1}+1)} = (-i)h_{j(2^{k-1}+j)} + ii = 0$, we get that $h_{j(2^{k-1}+j)} = i$, i.e., $j \rightarrow 2^{k-1} + j$; see Fig.4.6(b). Due to the connection of Q_{k-1} , we can show that $h_{2(2^{k-1}+2)} = h_{3(2^{k-1}+3)} = \dots = h_{2^{k-1}2^k} = i$. Then $S_2 = iI$ and $H = \begin{bmatrix} H(Q_{k-1}^{\phi_0}) & iI \\ -iI & H(Q_{k-1}^{-\phi_0}) \end{bmatrix} = H(Q_k^{\phi_0})$. Thus, there exists a switching matrix $D(\theta) = T$ such that $H(Q_k^{\phi_0}) = D(\theta)^{-1} H(Q_k^\phi) D(\theta)$. That is $Q_k^\phi \sim Q_k^{\phi_0}$.

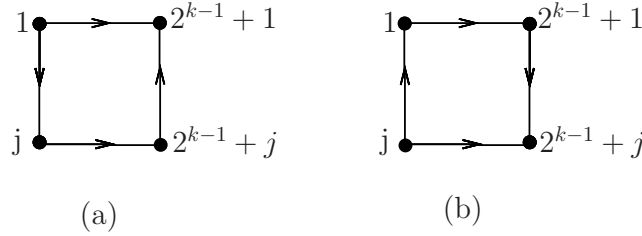


Figure 4.6: Two orientations of edges related to vertices $1, j, 2^{k-1} + 1, 2^{k-1} + j$.

Above all, we conclude that any optimum Hermitian energy oriented graph with underlying graph Q_k is unique (up to switching equivalence) for any positive integer k . The proof is complete. \blacksquare

Remark 4.1 The optimum Hermitian energy orientation ϕ_0 of Q_k is similar to the orientation obtained by Tian [13]. However, our proof is very different from Tian's. His proof is based on the skew adjacency matrix, while ours uses the third type mixed 4-cycle. Moreover, we prove that the optimum Hermitian energy orientations of Q_k are unique (up to switching equivalence).

References

- [1] C. Adiga, R. Balakrishnan, W. So, The skew energy of a digraph, *Linear Algebra Appl.* 432(2010), 1825–1835.
- [2] J. Bondy, U. Murty, *Graph Theory*, Springer, New York, 2008.
- [3] X. Chen, X. Li, H. Lian, 4-Regular oriented graphs with optimum skew energy, *Linear Algebra Appl.* 439(10)(2013), 2948–2960.

- [4] D. Cui, Y. Hou, On the skew spectra of Cartesian products of graphs, *Electron. J. Combin.* 20(2013), #P19.
- [5] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, London Mathematical Society Student Texts, vol. 75, Cambridge University Press, Cambridge, 2010.
- [6] S. Gong, G. Xu, 3-Regular digraphs with optimum skew energy, *Linear Algebra Appl.* 436(2012), 465–471.
- [7] S. Gong, G. Xu, W. Zhong, 4-regular oriented graphs with optimum skew energies, *Europ. J. Comb.* 36(2014), 77–85.
- [8] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sect. Forschungsz. Graz* 103(1978)1-22.
- [9] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.
- [10] X. Li, H. Lian, A survey on the skew energy of oriented graphs, Available at arXiv: 1304.5707.
- [11] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.
- [12] J. Liu, X. Li, Hermitian-adjacency matrices and Hermitian energies of mixed graphs, *Linear Algebra Appl.* 466(2015), 182–207.
- [13] G. Tian, On the skew energy of orientations of hypercubes, *Linear Algebra Appl.* 435(2011), 2140–2149.

